

## Membrane and fivebrane instantons from quaternionic geometry

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**ABSTRACT:** We determine the one-instanton corrections to the universal hypermultiplet moduli space coming both from Euclidean membranes and NS-fivebranes wrapping the cycles of a (rigid) Calabi-Yau threefold. These corrections are completely encoded by a single function characterizing a generic four-dimensional quaternion-Kähler metric without isometries. We give explicit solutions for this function describing all one-instanton corrections, including the fluctuations around the instanton to all orders in the string coupling constant. In the semi-classical limit these results are in perfect agreement with previous supergravity calculations.

**KEYWORDS:** Supergravity Models, Nonperturbative Effects.

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## 1. Introduction

Recently there has been considerable progress in understanding the quantum corrections to the hypermultiplet moduli spaces arising from compactifying type II strings on Calabi-Yau threefolds (CY<sub>3</sub>). These moduli spaces appear as sigma models for hypermultiplets in the  $N = 2, D = 4$  low-energy effective supergravity action. Since for type II strings the dilaton lives in a hypermultiplet, this sector is subject to both perturbative and non-perturbative stringy corrections.

In general the real dimension of the hypermultiplet moduli space is given by  $4n_H$  where the number of hypermultiplets is given by  $n_H = h_{1,2} + 1$  for type IIA and  $n_H = h_{1,1} + 1$  for type IIB, where  $h_{i,j}$  are the Hodge numbers of the  $CY_3$ . Local supersymmetry implies that the hypermultiplet moduli space has to be quaternion-Kähler [1], i.e., for  $n_H > 1$  the space has to be Einstein with holonomy contained in  $Sp(1) \times Sp(n_H)$ . The case  $n_H = 1$  is special. Here the definition of quaternion-Kähler geometry implies that the hypermultiplet moduli space has to be Einstein with self-dual Weyl curvature. This situation appears in type IIA compactifications on rigid  $CY_3$ , for which the Hodge number  $h_{1,2} = 0$ , and corresponds to the universal hypermultiplet.

At tree-level, the metric on the hypermultiplet moduli space can be determined either through the c-map [2–4] or by explicit dimensional reduction from ten dimensions [5, 6].

In perturbation theory, the hypermultiplet moduli space admits a number of commuting isometries which simplifies the description of the underlying quaternionic geometry. In that case, one can use the off-shell formulation for tensor multiplets which describe  $4n_H$ -dimensional quaternion-Kähler manifolds with  $n_H + 1$  commuting isometries [7]. This framework allowed to determine the perturbative one-loop corrections for generic Calabi-Yau compactifications [8], building on earlier work [9], and generalizing the case of the universal hypermultiplet [10, 11]. It was further argued in [8] that a non-renormalization theorem protects the hypermultiplet moduli space metric from higher loop corrections.

Non-perturbatively, there can be spacetime instanton corrections coming from wrapped Euclidean branes over the relevant cycles of the  $CY_3$  [12]. These will generically break (some of) the isometries present in perturbation theory. Beyond that, not much is known about their contribution to the hypermultiplet moduli space in the generic case with  $h_{1,2} \neq 0$ . This is due to our poor understanding of non-perturbative string theory, as well as our limited knowledge of the quaternion-Kähler geometry that underlies the hypermultiplet moduli space, see e.g. [13] for a general discussion. For the universal hypermultiplet, however, there are some partial results either for membrane instantons [14], or for NS-fivebrane instantons [15]. These results grew out of earlier work [16, 17, 11], see also [18–21]. Recently, these instanton corrections also played a prominent role in the construction of meta-stable de Sitter vacua [14, 22].

The aim of this paper is to determine the non-perturbative corrections to the universal hypermultiplet moduli space metric, including both membrane and fivebrane instantons. The key property for studying such corrections is that the corresponding metric must be quaternion-Kähler. Such metrics have been studied extensively in the context of Euclidean relativity and it was found in [23] that the general metric can be encoded in a single function satisfying a non-linear partial differential equation. We then find solutions to this equation which describe generic instanton corrections. As a non-trivial test, we reproduce the formulae for the instanton actions calculated in the supergravity approximation in [16, 17]. Moreover, our results are in perfect agreement with the explicit fivebrane instanton calculations done in the supergravity approximation [15]. In addition, for both membrane and fivebrane cases, we analyze the contributions from the fluctuations around the one-instanton to all orders in the string coupling constant. In particular, this includes the one-loop determinant around the instanton. It is remarkable that the constraints from

quaternionic geometry, combined with sensible boundary conditions, are restrictive enough to determine the form of the non-perturbative corrections that arise in string theory.

The rest of the paper is organized as follows. In section 2 we review the classical universal hypermultiplet moduli space and its perturbative corrections. In section 3 we present the metric for a generic four-dimensional quaternion-Kähler manifold in terms of a single function  $h$  and determine this function for the universal hypermultiplet and its perturbative corrections. Then, in section 4, we discuss the one-instanton corrections to the function  $h$ , together with the perturbative fluctuations around it. In particular we obtain explicit expressions for instanton corrections due to membranes and fivebranes and show their agreement with previously known results obtained in the semi-classical supergravity limit. We end with a brief discussion of our results in section 5. The technical details of our calculations can be found in the appendices.

## 2. The perturbative universal hypermultiplet

We start by reviewing the results for the perturbatively corrected universal hypermultiplet. This theory arises from compactifying type IIA strings on a rigid  $CY_3$ . The hypermultiplet sector of the 4-dimensional low energy effective Lagrangian can be obtained from dimensional reduction. At tree-level the relevant bosonic terms read

$$e^{-1}\mathcal{L}_T = -R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}e^{2\phi}H_\mu H^\mu - \frac{1}{2}e^{-\phi}(\partial_\mu\chi\partial^\mu\chi + \partial_\mu\varphi\partial^\mu\varphi) - \frac{1}{2}H^\mu(\chi\partial_\mu\varphi - \varphi\partial_\mu\chi). \quad (2.1)$$

Here  $\phi$  is the four-dimensional dilaton<sup>1</sup>,  $H^\mu = \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$  is the NS two-form field strength, and  $\varphi$  and  $\chi$  can be combined into a complex scalar  $C$  that descends from the holomorphic components of the RR 3-form with (complex) indices along the holomorphic 3-form of the  $CY_3$ . The NS two-form can then be dualized to an axion  $D$  by introducing a Lagrange multiplier

$$e^{-1}\mathcal{L}_{LM} = -H^\mu\partial_\mu D, \quad (2.2)$$

and eliminating  $H^\mu$  through its equation of motion. From the supergravity point of view it is often convenient to redefine this axion using the field

$$\sigma = D - \frac{1}{2}\chi\varphi. \quad (2.3)$$

In terms of  $\chi, \varphi, \sigma$  and  $r = e^\phi$  the classical universal hypermultiplet Lagrangian becomes

$$e^{-1}\mathcal{L}_{UHM}^{cl} = -R - \frac{1}{2r^2} \left[ (\partial_\mu r)^2 + r \left( (\partial_\mu\chi)^2 + (\partial_\mu\varphi)^2 \right) + (\partial_\mu\sigma + \chi\partial_\mu\varphi)^2 \right]. \quad (2.4)$$

The sigma model target space of the classical universal hypermultiplet is  $SU(2, 1)/U(2)$  [2, 3], with isometry group  $SU(2, 1)$ .

The perturbative corrections to the universal hypermultiplet have recently been obtained in [10]. There it was found that the Lagrangian (2.4) receives a non-trivial one-loop

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<sup>1</sup>Our conventions are such that the string coupling constant is related to the asymptotic value of the dilaton via  $g_s = e^{-\phi_\infty/2}$ .

correction while higher loop contributions can be absorbed by a coordinate transformation. The perturbatively corrected hypermultiplet metric then reads

$$ds_{\text{UHM}}^2 = \frac{1}{r^2} \left[ \frac{r+2c}{r+c} dr^2 + (r+2c)(d\chi^2 + d\varphi^2) + \frac{r+c}{r+2c} (d\sigma + \chi d\varphi)^2 \right]. \quad (2.5)$$

The constant  $c$  was obtained as

$$c = -\frac{4\zeta(2)\chi(X)}{(2\pi)^3} = -\frac{1}{6\pi}(h_{1,1} - h_{1,2}), \quad (2.6)$$

with  $\chi(X)$  denoting the Euler number of the  $\text{CY}_3$ , and one should set  $h_{1,2} = 0$  in our case.

At the perturbative level the hypermultiplet metric (2.5) retains four unbroken isometries. First of all there is a 3-dimensional Heisenberg group of isometries acting as shifts in  $\sigma$ ,  $\chi$  and  $\varphi$

$$r \rightarrow r, \quad \chi \rightarrow \chi + \gamma, \quad \varphi \rightarrow \varphi + \beta, \quad \sigma \rightarrow \sigma - \alpha - \gamma\varphi, \quad (2.7)$$

where  $\alpha, \beta$  and  $\gamma$  are real parameters. Additionally one has a rotational symmetry in the  $\chi$ - $\varphi$  plane parameterized by the real angle  $\delta$

$$r \rightarrow r, \quad D \rightarrow D, \quad \chi \rightarrow \chi \cos \delta - \varphi \sin \delta, \quad \varphi \rightarrow \chi \sin \delta + \varphi \cos \delta. \quad (2.8)$$

In this frame it is obvious that the isometry (2.8) corresponds to a phase transformation of the complex coordinate  $C = \frac{1}{2}(\chi + i\varphi)$  which, microscopically, is related to a rescaling of the Calabi-Yau holomorphic three-form by a phase. Notice that when working in terms of  $\sigma$  the isometry (2.8) acts non-trivial on  $\sigma$ :

$$\sigma \rightarrow \sigma + \chi\varphi \sin^2 \delta + \frac{1}{4}(\varphi^2 - \chi^2) \sin(2\delta). \quad (2.9)$$

Generically, one expects that (some of) these isometries will be broken by instanton corrections. For instance, fivebrane instantons will break the shift symmetry in  $\sigma$  to a discrete subgroup, whereas membrane instantons will break the shift symmetry in  $\varphi$  or  $\chi$ . We will discuss the fate of these isometries in more detail in section 4.

### 3. Four-dimensional quaternion-Kähler geometry

After having reviewed the perturbatively corrected universal hypermultiplet metric, we now proceed and introduce the Przanowski framework describing a general four-dimensional quaternion-Kähler metric. Subsequently we will recast the perturbative universal hypermultiplet in this framework and derive the linear partial differential equation governing small perturbations around the perturbative metric in subsections 3.2 and 3.3, respectively.

#### 3.1 The master equation

Four-dimensional quaternion-Kähler spaces coincide with Riemannian Einstein manifolds with a self-dual Weyl tensor. Such manifolds with a non-vanishing cosmological constant  $l$  were characterized in [24, 25, 23] in terms of solutions of a single differential equation. More

precisely, it has been shown that there always exists a system of local complex coordinates,  $z^1$  and  $z^2$ , such that the metric takes the form

$$ds^2 = g_{\alpha\bar{\beta}} (dz^\alpha \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^\alpha), \tag{3.1}$$

where  $\alpha, \beta = 1, 2$ ,  $z^{\bar{\beta}} = \bar{z}^\beta$  and

$$g_{\alpha\bar{\beta}} = -\frac{3}{l} \left( \partial_\alpha \partial_{\bar{\beta}} h + 2\delta_\alpha^2 \delta_{\bar{\beta}}^{\bar{2}} e^h \right). \tag{3.2}$$

This form for the metric components was also derived more recently in [26], where it was generalized to quaternion-Kähler spaces of higher dimensions. Notice that the metric is completely characterized in terms of a single real function  $h(z^\alpha, z^{\bar{\alpha}})$ . The constraints from quaternionic geometry imply [24] the following non-linear partial differential equation

$$\partial_1 \partial_{\bar{1}} h \cdot \partial_2 \partial_{\bar{2}} h - \partial_1 \partial_{\bar{2}} h \cdot \partial_{\bar{1}} \partial_2 h + (2\partial_1 \partial_{\bar{1}} h - \partial_1 h \cdot \partial_{\bar{1}} h) e^h = 0. \tag{3.3}$$

We call (3.3) the master equation. It describes in a compact way all four-dimensional quaternion-Kähler geometries.

Furthermore, in [23] this general result was specialized to the case of quaternion-Kähler metrics with one Killing vector field. It turns out that there are two distinct situations depending on the direction of the isometry. According to [23] there can be shifts along either

$$(A) \quad z^2 - z^{\bar{2}}, \quad \text{or} \quad (B) \quad z^1 - z^{\bar{1}}. \tag{3.4}$$

In both cases the metric is completely determined by the solutions of (3.3), which do not depend on the combination (3.4).

For the case (B), it was shown in [23] that the master equation (3.3) reduces to the three-dimensional Toda equation

$$\partial_z \partial_{\bar{z}} F + \partial_r^2 e^F = 0, \tag{3.5}$$

by means of the Lie-Bäcklund transformation

$$z = z^2, \quad \bar{z} = z^{\bar{2}}, \quad r = -(\partial_{z^1 + z^{\bar{1}}} h)^{-1}, \quad F = h - \log(\partial_{z^1 + z^{\bar{1}}} h)^2. \tag{3.6}$$

For a related discussion on such geometries, see also [27]. So all self-dual Einstein metrics possessing a Killing vector of type (B) can be characterized by solutions of the three-dimensional Toda equation. For the case (A) no such transformation is known explicitly, although the results of [27] imply this is still possible.<sup>2</sup>

It is somewhat surprising that the two classes in (3.4) have some physical meaning. For fivebrane instantons only, one can work in a framework in which the isometry of type (A) is manifest. In the case of membrane instantons only, i.e., in the absence of fivebrane charge, the isometry of type (B) is preserved. This follows from the results of the next subsection. The general situation is however when both types of instantons are present and hence no isometries will be preserved. This is the situation we describe in this paper, and we will therefore stick to the master equation (3.3) and its solutions.

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<sup>2</sup>We thank P. Tod for a discussion on this issue.

### 3.2 The universal hypermultiplet in the Przanowski framework

When using the Przanowski metrics above as target spaces in a low-energy supergravity action, supersymmetry requires to fix the value of the Ricci-scalar constructed from these metrics in terms of the gravitational coupling constant  $\kappa$ . In our conventions where we have chosen  $\kappa^{-2} = 2$ , the proper value is  $R = -6$ , implying that the value of the cosmological constant is fixed to  $l = -\frac{3}{2}$ .

To recast the perturbatively corrected universal hypermultiplet metric into Przanowski form, one should find a coordinate transformation which maps (2.5) to the metric (3.2) with the function  $h$  satisfying (3.3). This is achieved by setting

$$z^1 = \frac{1}{2}(u + i\sigma), \quad z^2 = \frac{1}{2}(\chi + i\varphi), \quad (3.7)$$

where

$$u = r - \frac{1}{2}\chi^2 + c \log(r + c). \quad (3.8)$$

The corresponding function  $h_0$  is given by

$$h_0 = \log(r + c) - 2 \log r, \quad (3.9)$$

and we have added the subscript 0 in order to indicate the perturbative solution. In this formula,  $r$  has to be understood as a function of  $z^1$  and  $z^2$ . The inverse transformation of (3.8) can only be done explicitly for  $c = 0$ . In that case, the classical solution is given by  $h_0 = -\log r$  with

$$c = 0 : \quad r = z^1 + \bar{z}^1 + \frac{1}{2}(z^2 + \bar{z}^2)^2. \quad (3.10)$$

It is easy to verify that this solves (3.3). To verify that the function (3.9) is a solution of (3.3) it is useful to first rewrite the master equation in terms of the real fields  $u, \sigma, \chi$  and  $\varphi$ ,

$$\begin{aligned} & (\partial_\chi^2 + \partial_\varphi^2)h (\partial_u^2 + \partial_\sigma^2)h - (\partial_\chi \partial_u h + \partial_\varphi \partial_\sigma h)^2 - (\partial_\chi \partial_\sigma h - \partial_\varphi \partial_u h)^2 \\ & + \left( 2(\partial_u^2 + \partial_\sigma^2)h - (\partial_u h)^2 - (\partial_\sigma h)^2 \right) e^h = 0. \end{aligned} \quad (3.11)$$

Changing basis again to the variables  $r, \sigma, \chi$  and  $\varphi$  using the identities

$$\partial_u = \frac{r+c}{r+2c} \partial_r, \quad \partial_\chi|_u = \partial_\chi|_r + \chi \frac{r+c}{r+2c} \partial_r, \quad \partial_u h_0 = -\frac{1}{r}, \quad \partial_\chi h_0 = -\frac{\chi}{r}, \quad (3.12)$$

it is then straightforward to check that (3.9) solves the master equation.

Moreover, note that  $h_0$  is a function of  $u$  and  $\chi$ , and thus of  $z^1 + \bar{z}^1$  and  $z^2 + \bar{z}^2$  only. Hence there are shift symmetries associated with both  $z^1 - \bar{z}^1$  and  $z^2 - \bar{z}^2$  such that the perturbative hypermultiplet metric fits simultaneously into both cases (3.4). In particular, this means that it can be described by a solution of the Toda equation (3.5). This fact was extensively used in [14] when studying membrane instanton corrections to the universal hypermultiplet. Performing the Lie-Bäcklund transformation for the solution (3.9), we obtain  $e^F = r + c$  which reproduces the result found in [14].

### 3.3 Deformations around the perturbative solution

After having established the Przanowski description of the perturbatively corrected universal hypermultiplet we now proceed and discuss the inclusion of non-perturbative corrections. In this course we will work in the region of the moduli space where the string coupling is small, so that these corrections are exponentially suppressed. In terms of the real variables this means that  $r$  and  $u$  are much larger than any other variable.

The solution describing the non-perturbative corrections to the universal hypermultiplet can be written as

$$h = h_0(u, \chi) + \Lambda(u, \sigma, \chi, \varphi) + \dots \quad (3.13)$$

Here  $h_0(u, \chi)$  is the perturbative solution (3.9) and  $\Lambda$  is an exponentially small correction. We assume that it encodes the one-instanton effects, whereas multi-instanton corrections are hidden in the dots. They will not be considered in the following and therefore we do not display them explicitly in our ansatz. We will therefore work in the linearized approximation and assume that the solution can be extended to a solution of the full non-linear equation by setting up an iteration scheme. For the case of the Toda equation, this was demonstrated in [14].

Utilizing (3.11) we can derive a partial differential equation for  $\Lambda$  which implements the constraints from quaternion-Kähler geometry. Substituting the ansatz (3.13) into (3.11) and expanding the result to the first order in  $\Lambda$  one finds

$$\begin{aligned} & (\partial_\chi^2 h_0 + 2e^{h_0}) (\partial_u^2 + \partial_\sigma^2) \Lambda + \partial_u^2 h_0 (\partial_\chi^2 + \partial_\varphi^2) \Lambda - 2\partial_\chi \partial_u h_0 (\partial_\chi \partial_u + \partial_\varphi \partial_\sigma) \Lambda \\ & - 2e^{h_0} \partial_u h_0 \partial_u \Lambda + e^{h_0} (2\partial_u^2 h_0 - (\partial_u h_0)^2) \Lambda = 0. \end{aligned} \quad (3.14)$$

For our purpose it is, however, more convenient to work directly in terms of the hypermultiplet variables. Therefore, we trade the variable  $u$  for  $r$  by means of (3.8). Then using the formulas (3.12), one can rewrite (3.14) as

$$\left[ \left( r + \chi^2 + 3c + \frac{c^2}{r+c} \right) \partial_\sigma^2 + (r+c) \partial_r^2 + \partial_\chi^2 + \partial_\varphi^2 - 2\chi \partial_\varphi \partial_\sigma + \left( 3 + \frac{2c}{r} \right) \partial_r + \frac{1}{r} \right] \Lambda = 0. \quad (3.15)$$

This equation is the master equation for the instanton corrections.

### 3.4 Solution generating technique

Before we discuss the solutions of (3.15) in the next section, let us first point out a general solution generating technique [28]. The master equation (3.3) has an invariance group of transformations, as one can easily verify,

$$z^1 \rightarrow \tilde{z}^1 = f(z^1, z^2), \quad z^2 \rightarrow \tilde{z}^2 = g(z^2), \quad (3.16)$$

where  $f$  and  $g$  are two arbitrary holomorphic functions. These transformations lead to new solutions of (3.3) of the form

$$h(z^1, z^2) \rightarrow \tilde{h}(z^1, z^2) = h\left(f(z^1, z^2), g(z^2)\right) - \log(g'(z^2)\bar{g}'(\bar{z}^2)), \quad (3.17)$$



where the prime stands for the derivative.

In general these new solutions lead to new metrics with different asymptotics. In our case, we want the new metric to have the same asymptotic behavior, namely the one given by (3.9). This puts certain constraints on  $f$  and  $g$ , and a class of functions which leave the boundary conditions invariant are the ones that generate the isometries. For instance, choosing

$$f(z^1, z^2) = z^1 - \gamma z^2 - \frac{1}{4}\gamma^2 - \frac{1}{2}i\alpha, \quad g(z^2) = z^2 + \frac{1}{2}(\gamma + i\beta), \quad (3.18)$$

generates the Heisenberg group of isometries (2.7), and similarly for the rotations (2.8). Clearly, the solution  $h_0$  as in (3.9) is invariant, and this guarantees that the asymptotic form remains the same. On the other hand, the instanton deformations  $\Lambda$  from (3.13) will not be invariant since some of the isometries from the Heisenberg group are broken non-perturbatively. However, at the level of the hypermultiplet metric the new solutions are related by a coordinate transformation. Thus, the solutions generated in this way do not produce physically inequivalent spaces.

This feature can be overcome when one works at the linearized level. It is clear that the perturbative isometries also generate new solutions of the linearized eq. (3.15). But in contrast to the non-linear case, this technique allows to generate new target spaces by taking linear combinations of the transformed instanton corrections. For example, having a solution  $\Lambda_0(r, \chi, \varphi, \sigma)$  of (3.15), the transformation  $\varphi \rightarrow \varphi + \beta$  generates the family of solutions

$$\Lambda_\beta(r, \chi, \varphi, \sigma) = \Lambda_0(r, \chi, \varphi + \beta, \sigma). \quad (3.19)$$

All of them are physically equivalent. But the superposition of instanton solutions of the form

$$\int d\beta C(\beta) \Lambda_\beta(r, \chi, \varphi, \sigma) \quad (3.20)$$

with any function  $C(\beta)$  having support on more than one point, generically leads to physically distinct target spaces.

These facts can be used to simplify the derivation of the general instanton solution, since they allow us to focus on one particular member of a family of solutions related by the isometry transformations. Once we find its exact form, all other solutions will follow by applying the perturbative isometry transformations and considering their linear combinations.

## 4. Instanton corrections to the universal hypermultiplet

### 4.1 Supergravity results about instanton actions

Before we start the calculation of instanton corrections to the universal hypermultiplet, let us briefly review the results obtained in the supergravity framework about the instanton actions [16, 17, 15].

There are two classes of instanton solutions, describing the wrapping of a NS-fivebrane or a membrane in type IIA string compactifications on a rigid  $CY_3$ . They produce corrections to the metric proportional to  $e^{-1/g_s^2}$  or  $e^{-1/g_s}$  respectively [12], and we repeat (see

footnote 1) that the relation with the dilaton is  $g_s = e^{-\phi_\infty/2}$ . These instantons can also be described as finite action solutions to the four-dimensional supergravity equations of motion for the universal hypermultiplet [16, 17], see also [19] for an earlier reference.

It was found that, for the NS-fivebrane instantons, the supergravity instanton action is given by<sup>3</sup>

$$S_{\text{inst}}^{(5)} = |Q_5| \left( \frac{1}{g_s^2} + \frac{1}{2} \chi^2 \right) + iQ_5 \sigma. \quad (4.1)$$

Here,  $\chi$  is treated as a coordinate on the moduli space, similarly to the dilaton. It is natural to interpret  $Q_5$  as the instanton number for the Euclidean NS-fivebrane, wrapped over the entire Calabi-Yau space. The  $\theta$ -angle like term  $e^{iQ_5 \sigma}$  breaks the isometry of the classical metric along  $\sigma$  to a discrete subgroup  $\mathbb{Z}$ . Notice further that the shift symmetry in  $\varphi$  is unbroken so, in the presence of fivebrane instantons only, this remains an isometry. Finally, the shift symmetry in  $\chi$  is explicitly broken.

The membrane instantons, as found in [16, 17, 15], can be parameterized by two charges and the instanton action is

$$S_{\text{inst}}^{(2)} = \left( |Q_2| + \frac{1}{2} |\chi Q_5| \right) \sqrt{\frac{4}{g_s^2} + \chi^2} + iQ_2 \varphi + iQ_5 \sigma. \quad (4.2)$$

For  $\chi = 0$  one obtains a more standard instanton action inversely proportional to the string coupling constant,  $S = 2|Q_2|/g_s$ , in which  $Q_2$  plays the role of the membrane charge. Notice that they also contain the fivebrane charge  $Q_5$ . The microscopic string theory interpretation of the additional terms proportional to  $\chi$  remains unclear. In the next subsections, we will show however that there exist solutions to the master equation (3.15) that reproduce these results exactly.

The fate of the isometries for membrane instantons differs from the fivebrane case, in the sense that the isometry along  $\varphi$  is also broken to a discrete subgroup. When  $Q_5$  is switched off, the shift along  $\sigma$  remains to be an isometry.

## 4.2 General constraints on instanton corrections

Our main goal is to determine the instanton corrections to the perturbative hypermultiplet metric (2.5) using the same strategy as in [14]. Namely, we find exponentially small corrections to the solution (3.9) of the master equation (3.3). These corrections in turn generate corrections to the metric, and by the results of [24, 25] the full metric will automatically satisfy the constraints of the quaternionic geometry. The main difference with [14], based on solutions of the Toda equation, is that we do not suppose the existence of an isometry since the combination of both membrane and fivebrane instantons does not preserve any continuous isometry.

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<sup>3</sup>Actually, in [16, 17] the instanton action was written in terms of  $\Delta\chi = \chi - \chi_0$  and  $\hat{\sigma} = \sigma + \chi_0 \varphi$ , where  $\chi_0$  is an arbitrary constant that was interpreted as a RR flux. It is clear that this result for the instanton action can be obtained from (4.1) by applying the shift isometry in  $\chi$  (2.7), with parameter  $\gamma = -\chi_0$ . Finding a solution of the master equation consistent with (4.1) is therefore sufficient. Similarly for the membrane instantons. We come back to this issue in subsection 4.3

Our goal is to solve the master equation (3.15) and consequently determine the instanton corrected hypermultiplet moduli space metric. Of course, this master equation possesses a large number of solutions. We are not interested in all of them but only in those which are physically appropriate. Therefore, we impose a set of requirements on the solution to be satisfied. Our conditions on admissible instanton corrections are the following:

1. For small string coupling  $g_s$  the instanton contributions should be exponentially suppressed.
2. In each instanton sector, there should be a perturbative series in  $g_s$  that describe the fluctuations around the instanton. These we represent as an (exponentiated) Laurent series, bounded from below and including the term  $\log g_s$ .
3. The shift symmetry in the NS scalar  $\sigma$  (or  $D$ ) should be broken by a theta-angle-like term only. This requirement is justified by the fact that the NS scalar comes from dualizing the NS two-form in four dimensions. The breaking of the isometry is triggered by the boundary term in the dualization process. This boundary term precisely generates the theta-angle that breaks the shift symmetry to a discrete subgroup.
4. For instanton solutions containing several charges, for instance the membrane and fivebrane instanton charges, the limit where one of the charges vanishes should still give rise to a regular solution which is exponentially suppressed.
5. The theta-angle terms should be independent of  $g_s$ . By this we mean that the purely imaginary terms in the exponent are independent of  $g_s$ .

Some of these requirements reflect a more stringent condition, namely that the full non-perturbative solution leads to a *regular* metric on the hypermultiplet moduli space. The perturbatively corrected metric corresponding to (3.9) develops a singularity at  $r = -c$ , and the instantons are supposed to resolve this singularity. This resolution can however not be understood in the one-instanton approximation we are working in. One would need to determine and sum up the entire instanton series to see how the singularity gets resolved. For an example of how this can work, we refer to [29].

An instanton correction which satisfies the first two conditions can be written in the following general form

$$\Lambda = A_0 r^\alpha \exp \left( - \sum_{k=1/2}^p f_k r^k + \sum_{k \in \mathbb{N}/2}^{\infty} \frac{A_k}{r^k} \right). \tag{4.3}$$

The terms containing the  $f_k$  are proportional to powers of the inverse string coupling constant  $g_s^{-1} = \sqrt{r}$ . They reflect the non-perturbative nature of the solution. The terms containing the  $A_k$  describe the fluctuations around the instanton. Usually, these are written in terms of a power series in  $g_s$  in front of the exponent, but they can be exponentiated as in (4.3) up to the term  $r^\alpha$  that would lead to a logarithm in the exponent. Here  $\alpha$  is some constant,  $p \in \mathbb{N}/2$ , and the sums over  $k$  run over (positive) integers and half-integers.

All the coefficients, including  $A_0$ , are complex functions of  $\chi$ ,  $\varphi$  and  $\sigma$ . The solution we construct is therefore complex, but we can add the complex conjugate to obtain a real solution. These two sectors will describe instantons and anti-instantons respectively. To leading order, which is the approximation we are working in, there is no mixing between these two sectors.

However, combining the conditions (3) and (5), one immediately concludes that all  $f_k$  and  $A_k$  except  $A_0$  must be real and  $\sigma$ -independent. We further study this generic ansatz in appendix A where we demonstrate that

$$\forall k > 1 \quad : \quad f_k = 0 . \tag{4.4}$$

We also prove there, in the linearized approximation we are working in, that there are no solutions with both  $f_1$  and  $f_{1/2}$  non-vanishing. Such terms might be generated however at subleading order, and would correspond to a combined system where both membrane and fivebrane instantons are present.

Thus, in accordance with the supergravity result, there are two classes of instanton corrections satisfying our conditions, which scale as  $e^{-f_1/g_s^2}$  and  $e^{-f_{1/2}/g_s}$ , respectively. These two classes are clearly related to NS-fivebrane and membrane instantons respectively [12]. As is known in string theory and will be confirmed by solving the master equation, in the first case the perturbative expansion around the instanton goes in even powers of  $g_s$ , whereas in the second case all integer powers contribute.

In the following we discuss the most general solutions, up to the action of the isometry group (see the end of section 3.3) of (3.15) satisfying all the above requirements which fit in these two classes.

### 4.3 Fivebrane instantons

In this subsection we study instanton corrections arising due to a Euclidean NS-fivebrane wrapping the entire Calabi-Yau. From the supergravity analysis it is known that the corresponding instanton action scales like  $g_s^{-2}$  and such corrections fit to our solutions with non-vanishing  $f_1$ . Since the perturbative expansion around such an instanton should go in even powers of  $g_s$ , one arrives at the following ansatz

$$\Lambda^{(5)} = A_0 r^\alpha \exp \left( -f_1 r + \sum_{k \in \mathbb{N}} \frac{A_k}{r^k} \right) . \tag{4.5}$$

In appendix B we solve the master equation (3.15) using the ansatz (4.5) and subject to the constraints listed above. We find the following *exact* solution<sup>4</sup>:

$$\Lambda^{(5)} = e^{\pm i Q_5 (\sigma + \frac{1}{2} \chi \varphi)} e^{-\frac{Q_5}{4} (\chi^2 + \varphi^2)} Z(r) . \tag{4.6}$$

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<sup>4</sup>In fact, we can find more general solutions where the function  $Z(r)$  is replaced by  $Z(r, \rho)$  with  $\rho = \chi^2 + \varphi^2$ . All these solutions have the same semi-classical behavior and differ in the subleading terms of the  $g_s$  expansion only. In the main text, we have focused on the simplest type of solution given by  $Z(r)$ . Some more details on the general case can be found in appendix B.

Here,  $Q_5$  is strictly positive,  $Q_5 > 0$ , and we denoted

$$Z(r) = \frac{C e^{Q_5 r}}{r(r+c)^{cQ_5}} \int_1^\infty e^{-2Q_5(r+c)t} \frac{dt}{t^{1+2cQ_5}}, \quad (4.7)$$

where  $C$  is some undetermined constant. Notice that this solution respects the U(1) isometry (2.8). Its leading term in the expansion in powers of  $g_s$  reads

$$\Lambda^{(5)} \approx C r^{-2-cQ_5} e^{\pm i Q_5 (\sigma + \frac{1}{2} \chi \varphi)} \exp \left[ -Q_5 \left( r + \frac{1}{4} (\chi^2 + \varphi^2) \right) \right]. \quad (4.8)$$

By expanding  $Z(r)$  to higher powers in  $1/r$ , one generates the loop expansion around the one-instanton sector to all orders in the string coupling constant.

By looking at the form of the exponent, we observe that this solution does not agree with the supergravity result (4.1). However, the instanton correction (4.6) is only one member of a family of solutions related by the perturbative isometry transformations. In particular, one can use the shift symmetry (2.7) to restore the isometry along  $\varphi$  which is manifest in (4.1). For this let us consider an instanton correction given by the following integral

$$\tilde{\Lambda}^{(5)} \equiv \sqrt{\frac{Q_5}{4\pi}} \int_{-\infty}^\infty \Lambda^{(5)}(r, \sigma, \chi, \varphi + \beta) d\beta = e^{\pm i Q_5 \sigma} e^{-\frac{1}{2} Q_5 \chi^2} Z(r). \quad (4.9)$$

The leading term of this solution now becomes

$$\tilde{\Lambda}^{(5)} \approx C r^{-2-cQ_5} e^{\pm i Q_5 \sigma} \exp \left[ -Q_5 \left( r + \frac{1}{2} \chi^2 \right) \right], \quad (4.10)$$

and shows precise agreement with the supergravity result (4.1) for the instanton action.

The isometry transformations allowing to generate new solutions can be applied as well to the solution (4.9). For example, using the  $\gamma$ -shift symmetry in  $\chi$  (2.7), one generalizes (4.9) to

$$\tilde{\Lambda}_{\chi_0}^{(5)} = e^{\pm i Q_5 (\sigma + \chi_0 \varphi)} e^{-\frac{Q_5}{2} (\Delta\chi)^2} Z(r), \quad (4.11)$$

where  $\Delta\chi = \chi - \chi_0$  and  $\chi_0$  is an arbitrary constant. This reproduces the instanton solution of [17] where  $\chi_0$  was interpreted as a RR-flux, see also footnote 3.

If instead one uses the rotation isometry (2.8), one obtains instanton corrections of the following form

$$\tilde{\Lambda}_\delta^{(5)} = e^{\pm i Q_5 (\sigma + \chi \varphi \sin^2 \delta + \frac{1}{4} (\varphi^2 - \chi^2) \sin(2\delta))} e^{-\frac{Q_5}{2} (\chi \cos \delta - \varphi \sin \delta)^2} Z(r). \quad (4.12)$$

Notice that for the particular angle  $\delta = \pi/2$  the correction (4.12) coincides with the one obtained by integrating (4.11) over  $\chi_0$

$$\tilde{\Lambda}_{\pi/2}^{(5)} = e^{\pm i Q_5 (\sigma + \chi \varphi)} e^{-\frac{Q_5}{2} \varphi^2} Z(r), \quad (4.13)$$

which is invariant under the  $\gamma$ -shift. On the other hand, integrating (4.12) with respect to  $\delta$ , one can restore the original U(1) symmetric solution (4.6). Indeed, it is easy to show that

$$\int_0^{2\pi} \tilde{\Lambda}_\delta^{(5)} \frac{d\delta}{2\pi} = e^{\pm i Q_5 (\sigma + \frac{1}{2} \chi \varphi)} e^{-\frac{Q_5}{4} (\chi^2 + \varphi^2)} Z(r) = \Lambda^{(5)}. \quad (4.14)$$

Thus, the solutions (4.9) and (4.6) generate two equivalent bases of instanton corrections and the integral transformations allow to pass from one to the other. Notice that the integral transform (4.14) maps the angle variable  $\sigma$  to the scalar  $D$  dual to the NS 2-form (see (2.3)).

It is instructive to discuss the fate of the perturbative isometries for each particular solution. Every  $\tilde{\Lambda}_\delta^{(5)}$  is invariant with respect to discrete shifts of  $\sigma$  and continuous shifts along a particular direction in the  $\chi$ - $\varphi$  plane (accompanied by a compensating transformation of  $\sigma$ ), whereas the rotation symmetry (2.8) is broken completely. In particular  $\tilde{\Lambda}^{(5)} = \tilde{\Lambda}_{\delta=0}^{(5)}$  preserves the shift symmetry in  $\varphi$ , and therefore falls into class (A) as defined in (3.4). In contrast the solution  $\Lambda^{(5)}$  as in (4.6) is symmetric in  $\chi$  and  $\varphi$  and respects the U(1) isometry corresponding to (2.8), whereas all continuous shifts are broken. Thus all our basis five-brane solutions preserve a residual symmetry group is  $\mathbb{Z} \times \mathbb{R}$  where the discrete factor corresponds to shifts of  $\sigma$  and  $\mathbb{R}$  comes from either the rotation or the shift symmetry in the  $\chi$ - $\varphi$  plane. Linear combinations of these basis solutions will, however, generically break this residual symmetry to  $\mathbb{Z}$ .

The question that now remains is which solution corresponds to the physically realized one in non-perturbative string theory. From our analysis, all solutions are on equal footing and preserve and break the same amount of isometries, although the anomalies appear in different sectors. However, these solutions yield different metrics and hence different low-energy effective actions. A possibility is that this also happens in string theory. From quantum field theory we know that, in the presence of more than one (classical) symmetries, it is possible to move anomalies from one current to another, depending on how one quantizes the theory and which regularization scheme is chosen. It is conceivable that such a mechanism also works in non-perturbative string theory. In that case, all our solutions could be considered as physically equivalent. It would be very interesting to understand this mechanism in more detail.

#### 4.4 Membrane instantons

Since in this case the instanton action scales like  $g_s^{-1}$ , we have the following ansatz

$$\Lambda^{(2)} = A_0 r^\alpha \exp \left( -f_{1/2} \sqrt{r} + \sum_{k \in \mathbb{N}/2} \frac{A_k}{r^k} \right). \tag{4.15}$$

The master equation (3.15) is solved in appendix C where the following two *exact* solutions are found

$$\Lambda_1^{(2)} = \frac{C}{r} e^{iQ_x \chi + iQ_\varphi \varphi} K_0(2Q_2 \sqrt{r+c}), \tag{4.16}$$

$$\begin{aligned} \Lambda_2^{(2)} = C' & \frac{\left( \sqrt{1 + \frac{\chi^2}{4(r+c)}} - \frac{\chi}{2\sqrt{r+c}} \right)^{2cQ_5}}{r \sqrt{4(r+c) + \chi^2}} e^{\pm iQ_2 \varphi \mp iQ_5 \sigma} \\ & \times \exp \left[ - \left( Q_2 + \frac{\chi}{2} Q_5 \right) \sqrt{4(r+c) + \chi^2} \right], \end{aligned} \tag{4.17}$$

where  $K_0$  is the modified Bessel function and we require  $Q_2$  and  $\chi Q_5$  to be strictly positive. In the first solution the charges are related by

$$Q_2^2 = Q_\chi^2 + Q_\varphi^2 . \tag{4.18}$$

The leading terms in the expansion in powers of  $g_s$  for the two instanton corrections are

$$\Lambda_1^{(2)} \approx \frac{C}{r} e^{iQ_\chi \chi + iQ_\varphi \varphi} \exp[-2Q_2 \sqrt{r}] , \tag{4.19}$$

$$\Lambda_2^{(2)} \approx C' r^{-3/2} e^{\pm iQ_2 \varphi \mp iQ_5 \sigma} \exp[-(2Q_2 + \chi Q_5) \sqrt{r}] . \tag{4.20}$$

By further expanding in powers of  $g_s$ , one generates the loop expansion around the membrane instanton. Notice that the two solutions have a different leading power of  $r$  in front of the exponent. This power is fixed by extending the leading order solution to the full one-instanton result, and depends crucially on the presence of  $Q_5$ .

The first solution (4.16) reproduces the leading behavior of the instanton corrections found in [12] and in [14]. It depends on two charges associated with both RR fields. However, based on the analysis [14], which goes beyond the linear order in  $\Lambda$ , one would expect that continuation of the solution to multi-instanton sectors will require setting one of the charges in (4.16) to zero.

The second solution (4.17) nicely coincides with the result (4.2) based on the supergravity analysis for the computation of the instanton action. It also depends on two charges,  $Q_2$  and  $Q_5$ , but their origin is different. For  $Q_5 = 0$ , the solution preserves the shift symmetry in  $\sigma$  and therefore belongs to class (B) in (3.4). For  $Q_5 \neq 0$ , we observe that the second charge gives rise to the factor  $e^{\pm iQ_5 \sigma}$ , which is the same as in the fivebrane instanton solution. Thus, it is natural to expect that the charge  $Q_5$  does have its origin in the NS-fivebrane, as was proposed in [17]. This fact also explains why the corrections including this  $Q_5$  charge were not found in [14]. The reason is that the application of the Toda equation requires the presence of the isometry in  $\sigma$ , whereas non-vanishing  $Q_5$  necessarily breaks it.

An interesting feature is that the instanton solution (4.20) is defined only for the phases (the theta-angle like terms) of a particular relative sign. Since for  $\chi > 0$  both  $Q_5$  and  $Q_2$  should be positive, the phases should be of different signs in this case. Correspondingly, in the opposite case of  $\chi < 0$ , the charges must have opposite sign and the phases are of the same sign. This indicates that only a configuration of an instanton and (anti-)instanton is stable for  $\chi < 0$  ( $\chi > 0$ ) and thus the stability depends on the sign of the RR field.

#### 4.5 Instanton corrected metric

Once  $\Lambda$  is found, one can determine the corrections to the hypermultiplet metric (2.5). In terms of the real coordinates (3.7) the Przanowski metric takes the form

$$ds^2 = (\partial_u^2 h + \partial_\sigma^2 h) (du^2 + d\sigma^2) + \left( \partial_\chi^2 h + \partial_\varphi^2 h + 2e^h \right) (d\chi^2 + d\varphi^2) + 2(\partial_\chi \partial_u h + \partial_\varphi \partial_\sigma h) (dud\chi + d\sigma d\varphi) + 2(\partial_\chi \partial_\sigma h - \partial_\varphi \partial_u h) (d\sigma d\chi - dud\varphi) . \tag{4.21}$$

Plugging  $u(r, \chi)$  from (3.8) and the instanton corrected function  $h$  from (3.13), and keeping only the linear terms in  $\Lambda$  one obtains the following result

$$ds^2 = ds_{\text{UHM}}^2 + ds_{\text{cor}}^2, \quad (4.22)$$

where the instanton correction to the hypermultiplet metric reads

$$\begin{aligned} ds_{\text{cor}}^2 = & \left( (f^{-1} \partial_r)^2 \Lambda + \partial_\sigma^2 \Lambda \right) (f^2 dr^2 + d\sigma^2) \\ & + \left( \partial_\chi^2 \Lambda + (\partial_\varphi - \chi \partial_\sigma)^2 \Lambda + f^{-1} \partial_r \Lambda + \frac{2(r+c)}{r^2} \Lambda \right) d\chi^2 \\ & + \left( (\partial_\chi + \chi f^{-1} \partial_r)^2 \Lambda + \partial_\varphi^2 \Lambda + \frac{2(r+c)}{r^2} \Lambda \right) d\varphi^2 \\ & + 2 (\partial_\chi \partial_r \Lambda + f (\partial_\varphi - \chi \partial_\sigma) \partial_\sigma \Lambda) dr d\chi \\ & + 2 \left( (\partial_\chi + \chi f^{-1} \partial_r) f^{-1} \partial_r \Lambda + \partial_\varphi \partial_\sigma \Lambda \right) d\sigma d\varphi \\ & + 2 (\partial_\chi \partial_\sigma \Lambda - f^{-1} (\partial_\varphi - \chi \partial_\sigma) \partial_r \Lambda) ((d\sigma + \chi d\varphi) d\chi - f dr d\varphi), \end{aligned} \quad (4.23)$$

and

$$f \equiv \frac{r+2c}{r+c}. \quad (4.24)$$

The complete hypermultiplet moduli space metric including membrane and fivebrane instantons is based on taking the sum  $\Lambda = \Lambda^{(5)} + \Lambda^{(2)}$ , at least in the one-instanton approximation.

We can also consider the cases of membranes and fivebrane instantons separately. In [15] fivebrane instanton corrections were explicitly computed using the four-dimensional effective supergravity action as a microscopic theory. Such an approach has of course its limitations, since one cannot compute the fluctuations around the instantons within supergravity. However, the results for the instanton action and correlation functions can be expected to give a reliable answer in the semi-classical approximation, with the one-loop determinants left unspecified. The metric obtained in this way was presented as

$$ds^2 = \frac{1}{r^2} dr^2 + \frac{1}{r} \left( (1-Y) d\chi^2 - 2i\tilde{Y} d\chi d\varphi + (1+Y) d\varphi^2 \right) + \frac{1}{r^2} (d\sigma + \chi d\varphi)^2, \quad (4.25)$$

where in our notations

$$Y = Y_+ + Y_-, \quad \tilde{Y} = Y_+ - Y_-, \quad Y_\pm = \frac{1}{16\pi^2} e^{\pm iQ_5\sigma} \left( S_{\text{inst}}^{(5)} \right)^2 K_{1\text{-loop}} e^{-S_{\text{inst}}^{(5)}}. \quad (4.26)$$

Here,  $S_{\text{inst}}^{(5)}$  is the real part of the fivebrane instanton action (4.1) and  $K_{1\text{-loop}}$  is the one-loop determinant which remained unknown.

To show that this result agrees with the one obtained in (4.23), let us choose in (4.23)  $\Lambda = \Lambda_+^{(5)} + \Lambda_-^{(5)}$  where  $\Lambda_\pm^{(5)}$  denote the fivebrane solutions (4.9) with plus and minus signs in the  $\sigma$ -dependent exponent, respectively. Then the instanton correction to the metric in the leading approximation is

$$\begin{aligned} ds_{\text{cor}}^2 \approx & 2Q_5 \left( \Lambda_+^{(5)} + \Lambda_-^{(5)} \right) (-d\chi^2 - (1 - 2Q_5\chi^2) d\varphi^2 + 2Q_5\chi (d\sigma d\varphi + dr d\chi)) \\ & - 2iQ_5\chi \left( \Lambda_+^{(5)} - \Lambda_-^{(5)} \right) (d\sigma d\chi - dr d\varphi + \chi d\chi d\varphi). \end{aligned} \quad (4.27)$$



It is easy to show that the following change of variables

$$\chi \rightarrow \chi + 2Q_5 r \chi \left( \Lambda_+^{(5)} + \Lambda_-^{(5)} \right), \quad \varphi \rightarrow \varphi + 2iQ_5 r \chi \left( \Lambda_+^{(5)} - \Lambda_-^{(5)} \right), \quad (4.28)$$

brings the metric (4.27) to the form

$$ds_{\text{cor}}^2 \approx 2Q_5 (2Q_5 \chi^2 - 1) \left[ \left( \Lambda_+^{(5)} + \Lambda_-^{(5)} \right) (d\varphi^2 - d\chi^2) - 2i \left( \Lambda_+^{(5)} - \Lambda_-^{(5)} \right) d\chi d\varphi \right]. \quad (4.29)$$

Upon identification

$$Y_{\pm} = 2Q_5 (2Q_5 \chi^2 - 1) r \Lambda_{\pm}^{(5)}, \quad (4.30)$$

one finds precise agreement with the instanton corrections to the metric in (4.25). Due to the leading behavior (4.10), the identification (4.30) is compatible with (4.26). Moreover, it allows us to read off the behavior of the one-loop determinant, up to a numerical constant,

$$K_{1\text{-loop}} = 64\pi^2 C g_s^{6+2cQ_5} \left( \chi^2 - \frac{1}{2Q_5} \right). \quad (4.31)$$

One can do a similar analysis for membrane instantons, by plugging in the solution (4.16) or (4.17) into the metric (4.23). Since in this case, we have no supergravity (or string theory) calculations to compare with, we refrain from giving explicit formulae.

## 5. Discussion

In this paper, we have used the constraints from quaternion-Kähler geometry to determine the structure of the membrane and NS-fivebrane instanton corrections to the universal hypermultiplet moduli space. As we have shown, the constraints reduce to solving a non-linear differential equation in terms of a single function. To describe the one-instanton corrections, including the perturbative fluctuations around it, it is sufficient to find solutions of the linearized differential equation, which is what we did in this paper. The solutions that we presented still contain undetermined integration constants. To fix these constants, one presumably needs to do a microscopic string theory calculation like the computation of the one-loop determinant around the membrane or fivebrane instanton.

To go beyond the one-instanton sector, one needs to solve the full non-linear differential equation. This can be done by setting up an iteration scheme similar to the one described in [14]. Most ideally the entire instanton series sums up to some special function respecting the symmetries of non-perturbative string theory compactified on Calabi-Yau threefolds. This is similar in spirit as to how modular functions in ten-dimensional IIB supergravity effective actions arise and determine the contributions from D-instantons by imposing  $SL(2, \mathbb{Z})$  symmetry non-perturbatively [30]. We leave this for future investigation.

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## A. Generic form of an instanton correction

This appendix discusses some generic properties of instanton corrections to the universal hypermultiplet which arise as a consequence of the master equation (3.15). In particular it is established that the master equation does not allow for solutions where the leading  $g_s$ -dependence of the instanton action is of the form  $S_{\text{inst}} \propto 1/g_s^n, n \geq 3$ . Furthermore it is shown that there are no solutions satisfying the “instanton conditions” (1) - (5) which simultaneously include non-vanishing  $1/g_s^2$  and  $1/g_s$ -terms in the instanton action. The later property implies that the three-charge instanton actions discussed in [19] do not give rise to supersymmetric corrections to the universal hypermultiplet moduli space.

Our starting point is the ansatz (4.3)

$$\Lambda = A_0 r^\alpha \exp \left( - \sum_{k=1/2}^p f_k r^k + \sum_{k \in \mathbb{N}/2}^{\infty} \frac{A_k}{r^k} \right), \quad (\text{A.1})$$

where, according to our general discussion,  $p = \mathbb{N}/2$  is finite and all  $f_k, A_k$  except  $A_0$  are real and  $\sigma$ -independent. We then substitute this ansatz into the master equation (3.15) and expand the resulting l.h.s. divided by  $\Lambda$  in inverse powers of  $r$ . This gives a system of differential equations on the coefficients  $f_k$  and  $A_k$ . Let us solve them one by one taking into account the conditions listed in section 4.2.

### A.1 Leading $g_s$ dependence of the instanton action

We start by proving that there are no solutions where the leading  $g_s$ -dependence of the instanton action is of the form  $S_{\text{inst}} \propto 1/g_s^n, n \geq 3$ .

Let us assume that  $p > 1$ . Then the first non-trivial equation appears at the order  $r^{2p}$  and reads

$$(\partial_\chi f_p)^2 + (\partial_\varphi f_p)^2 = 0. \quad (\text{A.2})$$

This implies that  $f_p$  is a constant. Taking this into account, one finds the next equation at the order  $r^{2p-1}$

$$p^2 f_p^2 + (\partial_\chi f_{p-\frac{1}{2}})^2 + (\partial_\varphi f_{p-\frac{1}{2}})^2 = 0. \quad (\text{A.3})$$

Since all  $f_k$  are real, the only solution of this equation is  $f_{p-\frac{1}{2}} = \text{const}$  and  $f_p = 0$ . By induction this implies that all  $f_k$  with  $k > 1$  must vanish.

The case  $k = 1$  is special. In this case eq. (A.2) still holds, implying that  $f_1$  is constant. The analog of (A.3), however, is modified according to

$$A_0^{-1} \partial_\sigma^2 A_0 + f_1^2 + (\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2 = 0, \quad (\text{A.4})$$

and has solutions for non-vanishing  $f_1$ . Thus the master equation restricts the ansatz for the instanton corrections to the form

$$\Lambda = A_0 r^\alpha \exp \left( - f_1 r - f_{1/2} \sqrt{r} + \sum_{k \in \mathbb{N}/2}^{\infty} \frac{A_k}{r^k} \right), \quad (\text{A.5})$$

with  $f_1$  being a positive constant.

## A.2 Absence of combined membrane and fivebrane instantons

Now we want to prove that there are no solutions satisfying our conditions (1) to (5) where  $f_1$  and  $f_{1/2}$  are both non-vanishing. As we already know,  $f_1$  must be a (positive) constant, say  $f_1 = |Q_5|$ . Then the general solution of (A.4) can be found by separation of variables and is based on

$$A_0(\chi, \varphi, \sigma) = \tilde{A}_0(\chi, \varphi) e^{iQ_5\sigma} + \hat{A}_0(\chi, \varphi) e^{-iQ_5\sigma}, \quad (\text{A.6})$$

where<sup>5</sup>

$$Q_\sigma = \sqrt{(Q_5)^2 + (\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2}. \quad (\text{A.7})$$

In the next step one establishes that the  $\sigma$ -independence in the coefficients  $A_k$ ,  $k > 0$  requires that  $Q_\sigma$  is constant. To this end one observes that when substituting the solution (A.6) into the equations arising at subleading powers in  $r$  these equations become polynomials in  $\sigma$  with  $\sigma$ -independent coefficients. In order for the solution to be consistent, all these coefficients have to vanish separately. In particular considering the coefficient multiplying the  $\sigma^2$ -term at order  $r^0$  one obtains:

$$(\partial_\chi ((\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2))^2 + (\partial_\varphi ((\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2))^2 = 0. \quad (\text{A.8})$$

As a result,  $f_{1/2}$  must satisfy

$$(\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2 = \tilde{Q}_5^2, \quad (\text{A.9})$$

where the constant  $\tilde{Q}_5^2$  is fixed by (A.4) as

$$\tilde{Q}_5^2 = Q_\sigma^2 - Q_5^2. \quad (\text{A.10})$$

Eq. (A.9) can be solved using the following trick. Introduce a function  $y(\chi, \varphi)$  such that

$$\partial_\chi f_{1/2} = \tilde{Q}_5 \cos y, \quad \partial_\varphi f_{1/2} = \tilde{Q}_5 \sin y. \quad (\text{A.11})$$

This solves (A.9) but the function  $y$  must satisfy the integrability condition

$$\cos y \partial_\chi y + \sin y \partial_\varphi y = 0. \quad (\text{A.12})$$

The general solution of this equation fits into two classes where  $y$  is either a non-trivial function of  $\chi$  and  $\varphi$  or simply a constant. We consider these two cases in turn.

### The case $y \neq \text{const}$

In this case solution of (A.12) can be written in an implicit form

$$\varphi \cos y - \chi \sin y = F(y), \quad (\text{A.13})$$

where  $F(y)$  is an arbitrary function. The corresponding function  $f_{1/2}$  is

$$f_{1/2} = 2Q_2 + \tilde{Q}_5 \left( \chi \cos y + \varphi \sin y - \int F(y) dy \right). \quad (\text{A.14})$$

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<sup>5</sup>In the following we restrict ourselves to  $\hat{A}_0(\chi, \varphi) = 0$ , which corresponds to considering perturbations due to instantons only. Anti-instantons can be added at any stage by requiring the reality of the solution.

Let us show that this solution is not consistent with our conditions on the instanton corrections. For this we go to the next constraint, which arises at the order  $r^{1/2}$ :

$$(|Q_5| - \partial_\chi^2 - \partial_\varphi^2)f_{1/2} - 2(\partial_\chi f_{1/2} \partial_\chi + \partial_\varphi f_{1/2} \partial_\varphi) \log A_0 + 2iQ_\sigma \chi \partial_\varphi f_{1/2} = 0. \quad (\text{A.15})$$

First note that for  $\tilde{Q}_5 = 0$ ,  $f_{1/2} = 2Q_2$  is just a constant and (A.15) requires  $Q_2 = f_{1/2} = 0$ . This is, however, not consistent with our requirement that both  $f_1$  and  $f_{1/2}$  are non-zero. Thus we take  $\tilde{Q}_5 \neq 0$  in the following. In this case eq. (A.15) can be used to determine  $A_0$ .

It is convenient to use the independent variables  $y$  and  $\chi$  instead of  $\varphi$  and  $\chi$ . Using eqs. (A.11), (A.14) and (A.13) it then follows that

$$2\tilde{Q}_5 \cos y \partial_\chi \log A_0 = 2|Q_5|Q_2 + \tilde{Q}_5|Q_5| \left( \frac{\chi}{\cos y} + F(y) \tan y - \int F(y) dy \right) + 2i\tilde{Q}_5 Q_\sigma \chi \sin y - \frac{\tilde{Q}_5 \cos y}{\chi + F(y) \sin y + F'(y) \cos y}, \quad (\text{A.16})$$

where the derivative with respect to  $\chi$  is taken at constant  $y$ . From this result one immediately concludes that the three charges  $Q_5$ ,  $\tilde{Q}_5$  and  $Q_2$  cannot all be independent. To avoid singularities at vanishing  $\tilde{Q}_5$ ,  $Q_2 Q_5$  should either vanish or be proportional to  $\tilde{Q}_5$ . Assuming that one of this situations is realized, one finds  $A_0$

$$A_0 = \frac{A(y) e^{iQ_\sigma(\sigma + \frac{1}{2} \chi^2 \tan y)}}{(\chi + F \sin y + F' \cos y)^{1/2}} \exp \left[ \frac{\tilde{Q}_2 \chi}{\cos y} + \frac{|Q_5| \chi}{2 \cos y} \left( \frac{\chi}{2 \cos y} + \tilde{F}(y) \right) \right], \quad (\text{A.17})$$

where  $A(y)$  is undetermined function and we introduced  $\tilde{Q}_2 = \frac{Q_2 |Q_5|}{Q_5}$  and

$$\tilde{F}(y) = F(y) \tan y - \int F(y) dy. \quad (\text{A.18})$$

To make further conclusions, one has to consider the equation at the order  $r^0$

$$-Q_\sigma^2 (\chi^2 + 3c) + \frac{1}{4} f_{1/2}^2 + cQ_5^2 - 2\alpha|Q_5| - 2(\partial_\chi f_{1/2} \partial_\chi + \partial_\varphi f_{1/2} \partial_\varphi) A_{1/2} + A_0^{-1} (\partial_\chi^2 + \partial_\varphi^2) A_0 - 2iQ_\sigma \chi \partial_\varphi \log A_0 - 3|Q_5| = 0. \quad (\text{A.19})$$

From this equation one can determine the coefficient  $A_{1/2}$ . However, since  $A_{1/2}$  is real, the imaginary part of the equation imposes conditions on the functions introduced above. One can show that for non-vanishing  $Q_\sigma$  ( $Q_\sigma = 0$  leads to  $\tilde{Q}_5 = Q_5 = 0$ ) it vanishes only if<sup>6</sup>

$$A(y) = C \sqrt{\cos y} \exp \left( \frac{|Q_5|}{4} \tilde{F}^2(y) + \tilde{Q}_2 \tilde{F}(y) - \frac{iQ_\sigma}{2} \left( F^2 \tan y + \int ((F')^2 - F^2) dy \right) \right). \quad (\text{A.21})$$

---

<sup>6</sup>For a special solution

$$F(y) = \varphi_0 \cos y - \chi_0 \sin y, \quad (\text{A.20})$$

where  $\chi_0, \varphi_0$  are some constants, there is additional possibility to add in the exponent of (A.21) the term  $iQ_y y$  with  $Q_y$  being a constant. However, such a term does not affect any further conclusions.

Plugging this result into (A.17), one finds

$$A_0 = \frac{C e^{iQ_\sigma(\sigma + \frac{1}{2}(\tan y(\chi^2 - F^2) - \int((F')^2 - F^2)dy))}}{\left(\frac{\chi}{\cos y} + F \tan y + F'\right)^{1/2}} e^{\tilde{Q}_2\left(\frac{\chi}{\cos y} + \tilde{F}\right) + \frac{|Q_5|}{4}\left(\frac{\chi}{\cos y} + \tilde{F}\right)^2}. \quad (\text{A.22})$$

One observes that due the sign of the last term in the exponent of  $A_0$  the instanton correction diverges in the region where  $\frac{\chi}{\cos y} + \tilde{F}$  is large. Such a region can be achieved, for example, by keeping  $y$  fixed and considering large  $\chi$ . Therefore, we are in contradiction with condition 1 from our list. This implies that  $Q_5$  must vanish in agreement with our statement that either  $f_1$  or  $f_{1/2}$  vanish.

### The case $y = \text{const}$

There still remains a possibility to have another solution of (A.12) which is  $y = \text{const}$ . It gives

$$f_{1/2} = 2Q_2 + \tilde{Q}_5(\chi \cos y + \varphi \sin y). \quad (\text{A.23})$$

However, one can use the rotation isometry (2.8) and the comment in the end of section 4.2 to put  $y = 0$ . Thus, in this case it is enough to consider

$$f_{1/2} = 2Q_2 + \tilde{Q}_5\chi. \quad (\text{A.24})$$

Notice that it fits to the more general solution (A.14) where however  $y$  should be considered as a constant instead of to be determined by (A.13).

To investigate this type of solution further, one can proceed as in the previous case. Then eq. (A.15) and regularity assumptions again imply that either  $Q_2Q_5$  vanish or  $\sim \tilde{Q}_5$  and

$$A_0 = C e^{iQ_\varphi\varphi} e^{iQ_\sigma\sigma + \tilde{Q}_2\chi + \frac{1}{4}|Q_5|\chi^2}. \quad (\text{A.25})$$

The  $\varphi$ -dependence thereby follows from the vanishing of the imaginary part of (A.19), yielding  $A(\varphi) = C e^{iQ_\varphi\varphi}$ . Thus, again for non-vanishing  $Q_5$  the solution diverges for large  $\chi$  and should be discarded. As a result, only instanton corrections where either  $f_1$  or  $f_{1/2}$  are non-vanishing are admissible.

## B. Solution of the master equation in the five-brane case

We now proceed and solve eq. (3.15) with boundary conditions corresponding to a five-brane instanton. In order to arrive at the exact solution (4.6) we thereby follow a two step procedure. In the first step we substitute the ansatz (4.5) into the master equation for instanton corrections and expand the resulting expression in inverse powers of  $r$ . Equating the coefficients in this expansion to zero leads to partial differential equations for the functions  $A_k$  which can be solved order by order. Thereby it turns out that the dependence of the  $A_k$  on  $\varphi$  and  $\chi$  can be deduced from the first few orders in the perturbative expansion. This knowledge allows to refine the ansatz (4.5) in such a way that it results in a partial differential equation governing the dilaton dependence of the solution. The solution of this equation then gives rise to the exact one-instanton correction which includes all orders of perturbation theory around the instanton.

We start by adapting the results of the previous section to the case of fivebrane instantons by setting  $f_{1/2} = 0$ . Eq. (A.4) then implies that <sup>7</sup>

$$f_1 = |Q_5|, \quad A_0 = \tilde{A}_0(\chi, \varphi) e^{iQ_5\sigma}. \quad (\text{B.1})$$

The function  $\tilde{A}_0$  is determined by eq. (A.19) with  $f_{1/2} = 0$  and  $Q_\sigma = Q_5$ :

$$(\partial_\chi^2 + \partial_\varphi^2)\tilde{A}_0 - 2iQ_5\chi\partial_\varphi\tilde{A}_0 - (\chi^2 + 2c)Q_5^2\tilde{A}_0 - (2\alpha + 3)|Q_5|\tilde{A}_0 = 0. \quad (\text{B.2})$$

One class of solutions to this equation can be found by substituting  $\tilde{A}_0^\pm = e^{\pm\frac{1}{2}Q_5\chi^2}\hat{A}_0^\pm$  and passing to complex coordinates  $z = \frac{1}{2}(\chi + i\varphi)$ . The partial differential equations determining  $\hat{A}_0^\pm(z, \bar{z})$  then become

$$\begin{aligned} + : \partial_z\partial_{\bar{z}}\hat{A}_0^+ + 2Q_5(z + \bar{z})\partial_z\hat{A}_0^+ - \kappa^+\hat{A}_0^+ &= 0, \\ - : \partial_z\partial_{\bar{z}}\hat{A}_0^- - 2Q_5(z + \bar{z})\partial_{\bar{z}}\hat{A}_0^- - \kappa^-\hat{A}_0^- &= 0, \end{aligned} \quad (\text{B.3})$$

with

$$\kappa^\pm = 2c(Q_5)^2 + (2\alpha + 3)|Q_5| \mp Q_5. \quad (\text{B.4})$$

Both equations can be solved by separation of variables. The separable solutions are labelled by a complex eigenvalue  $\lambda$  and are given by

$$\begin{aligned} \hat{A}_{0,\lambda}^+ &= C(\lambda + 2Q_5z)^{\frac{\kappa_+}{2Q_5}} e^{-Q_5\bar{z}^2 + \lambda\bar{z}}, \\ \hat{A}_{0,\lambda}^- &= C(\lambda - 2Q_5\bar{z})^{-\frac{\kappa_-}{2Q_5}} e^{Q_5z^2 + \lambda z}. \end{aligned} \quad (\text{B.5})$$

Since the functions  $\tilde{A}_0^\pm$  resulting from (B.5) have different asymptotics at large  $\chi$  and  $\varphi$  they provide linearly independent solutions. The general solution of (B.2) is then given by linear combinations of these two families

$$A_0 = e^{iQ_5\sigma} \int d^2\lambda \left( a_+(\lambda) e^{\frac{1}{2}Q_5\chi^2} \hat{A}_{0,\lambda}^+(\chi, \varphi) + a_-(\lambda) e^{-\frac{1}{2}Q_5\chi^2} \hat{A}_{0,\lambda}^-(\chi, \varphi) \right), \quad (\text{B.6})$$

with arbitrary functions  $a_\pm(\lambda)$ . For large values of  $\chi$  and  $\varphi$  only one of the branches in (B.6) is bounded. Which of the two branches is regular thereby depends on the sign of  $Q_5$  and is given by  $A_0^-$  ( $A_0^+$ ) for positive (negative)  $Q_5$ , respectively. The instanton solution consists of the bounded term only

$$A_0 = e^{iQ_5\sigma} \int d^2\lambda a(\lambda) (\lambda - |Q_5|(\chi - i\epsilon\varphi))^{-\kappa} e^{\frac{i}{2}\varphi(Q_5\chi + \epsilon\lambda)} e^{-\frac{|Q_5|}{4}(\chi^2 + \varphi^2) + \frac{1}{2}\chi\lambda}, \quad (\text{B.7})$$

where

$$\epsilon = \text{sign}(Q_5), \quad \kappa = 2 + \alpha + c|Q_5|. \quad (\text{B.8})$$

Note that the parameter  $\lambda$  appearing in the solution (B.7) can be generated by the solution generating technique discussed in subsection 3.4. To illustrate this, we start from the solution  $A_0$  with  $a(\lambda) \propto \delta(\lambda)$ ,

$$A_0 = C(\chi - i\epsilon\varphi)^{-\kappa} e^{iQ_5(\sigma + \frac{1}{2}\chi\varphi)} e^{-\frac{1}{4}|Q_5|(\chi^2 + \varphi^2)}, \quad (\text{B.9})$$

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<sup>7</sup>In principle one should also include a term proportional to  $\exp(-iQ_5\sigma)$  in  $A_0$ . Since the final solution for  $\Lambda$  has to be real, these terms can be restored by adding the complex conjugate of the solution found from  $A_0$  given below.

with  $C$  being a constant. The  $\lambda$ -dependent solution can then be obtained by applying the  $\beta$  and  $\gamma$  shifts, eq. (2.7), to (B.9) and identifying  $\lambda = -|Q_5|(\gamma - i\epsilon\beta)$ . Superposing the corresponding solutions with a suitable measure yields the general solution (B.7). Based on this observation, we can then simplify our further analysis by working with the particular solution (B.9) in the following.

To get some insights on the field dependence of the subleading coefficients  $A_k(\chi, \varphi)$ ,  $k \geq 1$  we pass to the next order in the expansion and consider the coefficient at order  $r^{-1}$ . Equating this coefficient to zero results in a partial differential equation for  $A_1(\chi, \varphi)$

$$(\partial_\chi^2 + \partial_\varphi^2)A_1 + 2(\partial_\chi \log A_0 \partial_\chi + (\partial_\varphi \log A_0 - iQ_5\chi) \partial_\varphi) A_1 + 2|Q_5|A_1 - c^2Q_5^2 - 2c(\alpha + 1)|Q_5| + (\alpha + 1)^2 = 0. \quad (\text{B.10})$$

This equation is solved by substituting  $A_0$  from (B.9) and splitting the resulting complex equation into its real and imaginary part. Taking into account that  $A_k(\chi, \varphi)$ ,  $k \geq 1$  are real the imaginary part yields

$$(Q_5 - 2\kappa\epsilon(\chi^2 + \varphi^2)^{-1})(\varphi\partial_\chi - \chi\partial_\varphi)A_1(\chi, \varphi) = 0. \quad (\text{B.11})$$

This implies that  $A_1(\chi, \varphi) = A_1(\rho)$ , with  $\rho = \chi^2 + \varphi^2$ . Rewriting the real part of (B.10) in terms of  $\rho$  gives an ordinary differential equation for  $A_1(\rho)$

$$4\rho\partial_\rho^2 A_1 - 2(\rho|Q_5| + 2\kappa - 2)\partial_\rho A_1 + 2|Q_5|A_1 - c^2Q_5^2 - 2c(\alpha + 1)|Q_5| + (\alpha + 1)^2 = 0, \quad (\text{B.12})$$

which has the solution

$$A_1(\rho) = (2\kappa - 2 + |Q_5|\rho) \left[ C_1 + C_2 \int d\rho \frac{\rho^{\kappa-1} e^{\frac{1}{2}|Q_5|\rho}}{(2\kappa - 2 + |Q_5|\rho)^2} \right] - \frac{1}{2|Q_5|} (c^2Q_5^2 + 2c(\alpha + 1)|Q_5| - (\alpha + 1)^2). \quad (\text{B.13})$$

In the limit  $Q_5 \rightarrow 0$  the solution  $A_1(\rho)$  is regular for  $\alpha = -1$  only. But since  $Q_5$  is the only instanton charge of the solution, and the instanton “does not exist” if  $Q_5 = 0$  we do not insist on the regularity of  $A_1(\rho)$  at this point in accord with condition (4). In fact we will argue below that the physical instanton solution should correspond to  $\kappa = 0$  and, by virtue of (B.8), to  $\alpha = -2 - c|Q_5|$ .

At higher orders of the expansion the pattern encountered for  $A_1(\chi, \varphi)$  repeats itself. The partial differential equations which determine the functions  $A_k(\chi, \varphi)$  for  $k \geq 2$  again split into a real and imaginary part. The later has to vanish independently at every order in the expansion and is given by

$$(Q_5 - 2\kappa\epsilon(\chi^2 + \varphi^2)^{-1})(\varphi\partial_\chi - \chi\partial_\varphi)A_k(\chi, \varphi) = 0. \quad (\text{B.14})$$

Based on these equations we conclude that

$$A_k(\chi, \varphi) = A_k(\rho), \quad k \geq 1. \quad (\text{B.15})$$

This result motivates the following refined ansatz for the five-brane instanton corrections

$$\begin{aligned} \Lambda^{(5)} &= (\chi - i\epsilon\varphi)^{-\kappa} e^{iQ_5(\sigma + \frac{1}{2}\chi\varphi)} e^{-\frac{1}{4}|Q_5|\rho} Z(r, \rho) \\ &= \rho^{-\kappa/2} e^{i\epsilon\kappa \arctan(\varphi/\chi)} e^{iQ_5(\sigma + \frac{1}{2}\chi\varphi)} e^{-\frac{1}{4}|Q_5|\rho} Z(r, \rho). \end{aligned} \tag{B.16}$$

Substituting this ansatz into the master equation (3.15) gives rise to a partial differential equation for the unknown function  $Z(r, \rho)$

$$\begin{aligned} &\left[ (r+c)\partial_r^2 + 4\rho\partial_\rho^2 + \left(3 + \frac{2c}{r}\right)\partial_r - 2(|Q_5|\rho + 2\kappa - 2)\partial_\rho \right. \\ &\quad \left. - Q_5^2 \left( r + 3c + \frac{c^2}{r+c} \right) - |Q_5|(1 - 2\kappa) + \frac{1}{r} \right] Z(r, \rho) = 0. \end{aligned} \tag{B.17}$$

The general solution can be found by separation of variables. We will however focus on a particular class of solutions, namely those for which  $Z(r, \rho) = Z(r)$  is independent of  $\rho$ . In terms of the perturbative expansion this corresponds to setting the integration constants  $C_i$  appearing in the coefficients  $A_k(\rho)$  to zero (cf. (B.13)). In this case (B.17) simplifies to an ordinary differential equation for  $Z(r)$ . Setting

$$Z(r) = r^{-1} (r+c)^{|Q_5|} \tilde{Z}(\xi(r)), \quad \xi = r+c, \tag{B.18}$$

the resulting equation for  $\tilde{Z}(\xi)$  takes the form

$$\left( \partial_\xi^2 + \frac{q}{\xi} \partial_\xi - Q_5^2 - |Q_5| \frac{q-2\kappa}{\xi} \right) \tilde{Z}(\xi) = 0, \tag{B.19}$$

with  $q = 1 + 2c|Q_5|$ . For non-zero values of  $\kappa$  the solution to this equation is given by Whittaker functions (cf. *e.g.*, AS).

The case  $\kappa = 0$  is special as in this case the  $\theta$ -angle like term  $\propto \exp(i\epsilon\kappa \arctan(\varphi/\chi))$  in (B.16) is absent. While such terms do not lead to a violation of the conditions (1) - (5) stated in subsection 4.2, it seems very unlikely that the five-brane instanton solution actually contains such angle-terms. Therefore we will focus on  $\kappa = 0$  in the following. In this case the general solution of (B.19) is

$$\tilde{Z}(\xi) = C_1 e^{|Q_5|\xi} + C_2 (2|Q_5|)^{q-1} e^{|Q_5|\xi} \Gamma(1-q, 2|Q_5|\xi), \tag{B.20}$$

where  $\Gamma(p, x) = \int_x^\infty e^{-t} t^{p-1} dt$  is the incomplete gamma function. The first term in (B.20) increases exponentially in  $\xi = r+c$  and violates condition (1) while the second term indeed gives rise to a solution which is exponentially suppressed for large values of the dilaton. Consequently we set  $C_1 = 0$  while keeping  $C_2 = C$  as a free parameter. The resulting five-brane instanton correction is then given by

$$\Lambda^{(5)} = \frac{C}{r(r+c)^{|Q_5|}} e^{iQ_5(\sigma + \frac{1}{2}\chi\varphi)} e^{|Q_5|(r - \frac{1}{4}(\chi^2 + \varphi^2))} \int_1^\infty e^{-2|Q_5|(r+c)t} \frac{dt}{t^{1+2c|Q_5|}}. \tag{B.21}$$

This result completes the derivation of the  $\Lambda^{(5)}$  given in eq. (4.6).



### C. Solution of the master equation in the membrane case

After discussing the five-brane instanton corrections associated with the instanton action (4.1) we now proceed with analyzing solutions of (3.15) corresponding to membrane instantons. In order to obtain exact (linearized) solutions we again follow the two step procedure of the previous section, i.e., we first carry out a perturbative analysis in  $g_s$  to gain some insights on the field dependence of the coefficient functions  $A_k(\chi, \varphi)$ . This information is then used to refine the ansatz for the membrane case. Substituting this improved ansatz into (3.15) then gives rise to a (partial) differential equation from which the exact solution can be determined. This program will lead to two types of membrane instanton corrections, the two charge membrane instanton corrections discussed in [14] and the membrane instanton with five-brane charge associated with the instanton action (4.2).

The starting point of the analysis are again the results obtained in appendix A. These are adapted to the membrane case by imposing

$$f_1 = |Q_5| = 0, \quad A_0 = \tilde{A}_0(\chi, \varphi)e^{iQ_\sigma\sigma} \tag{C.1}$$

with

$$Q_\sigma^2 = (\partial_\chi f_{1/2})^2 + (\partial_\varphi f_{1/2})^2. \tag{C.2}$$

Eqs. (C.1) and (C.2) give rise to two distinct classes of solutions corresponding to  $Q_\sigma = 0$  and  $f_{1/2}$  being a positive constant and  $Q_\sigma \neq 0$  which leads to a field dependent leading coefficient  $f_{1/2}$ . We will now discuss these possibilities in turn.

#### C.1 Two-charge membrane instantons

We first consider the case  $Q_\sigma = 0$ . Eq. (C.2) then implies that

$$f_{1/2} = 2|Q_2| \tag{C.3}$$

is a positive constant. Substituting this result into eq. (A.15), one finds that the coefficient equation at order  $r^{1/2}$  does not impose further restrictions. The equation determining  $\tilde{A}_0(\chi, \varphi)$  arises at order  $r^0$ . Substituting (C.3) into (A.19) it becomes

$$(\partial_\chi^2 + \partial_\varphi^2) \tilde{A}_0 + Q_2^2 \tilde{A}_0 = 0. \tag{C.4}$$

At this stage it suffices to consider special solutions of the form

$$\tilde{A}_0 = (a_0 + \tilde{a}_0 \eta) e^{iQ_\chi\chi + iQ_\varphi\varphi}, \quad Q_\chi^2 + Q_\varphi^2 = Q_2^2, \tag{C.5}$$

with

$$\eta \equiv Q_\varphi\chi - Q_\chi\varphi, \tag{C.6}$$

and arbitrary coefficients  $a_0, \tilde{a}_0$ . The general solution of (C.4) is then obtained by superimposing the solution (C.5) for different values of the charges  $Q_\chi, Q_\varphi$  associated with the two RR fields.

In order to get some information on the functions  $A_{k/2}(\chi, \varphi)$ ,  $k \geq 1$  we consider the coefficient function at order  $r^{-1/2}$ . Using the results (C.3) and (C.5) together with the

reality of  $A_{1/2}$  one obtains two equations corresponding to the real and imaginary part of the coefficient multiplying  $r^{-1/2}$ . The imaginary part gives rise to the equation

$$(Q_\chi \partial_\chi + Q_\varphi \partial_\varphi) A_{1/2} = 0, \quad (\text{C.7})$$

which indicates that

$$A_{1/2}(\chi, \varphi) = A_{1/2}(\eta) \quad (\text{C.8})$$

is a function of  $\eta$  only. Using this result, the real part equation becomes a ordinary differential equation for  $A_{1/2}(\eta)$

$$(a_0 + \tilde{a}_0 \eta) (Q_2^2 \partial_\eta^2 A_{1/2} - \frac{1}{2} |Q_2| (4\alpha + 5)) + 2 \tilde{a}_0 Q_2^2 \partial_\eta A_{1/2} = 0. \quad (\text{C.9})$$

This equation is readily solved and one obtains

$$A_{1/2}(\eta) = \frac{1}{2|Q_2|} \frac{4\alpha + 5}{a_0 + \tilde{a}_0 \eta} \left( \frac{1}{6} \tilde{a}_0 \eta^3 + \frac{1}{2} a_0 \eta^2 + a_{1/2} \eta + \tilde{a}_{1/2} \right). \quad (\text{C.10})$$

Here  $a_{1/2}$  and  $\tilde{a}_{1/2}$  are real integration constants.

This pattern repeats itself for  $A_{k/2}(\chi, \varphi)$ ,  $k \geq 2$ . The partial differential equation determining  $A_{k/2}$  decomposes into real and imaginary part. The equation arising from the imaginary part shows that  $A_{k/2}(\chi, \varphi) = A_{k/2}(\eta)$ . Thus we conclude that the functions  $A_{k/2}(\chi, \varphi)$ ,  $k \geq 1$  depend on  $\chi, \varphi$  through the combination (C.6) only.

This result motivates refining the ansatz (4.3) to the form

$$\Lambda_1^{(2)} = e^{iQ_\chi \chi + iQ_\varphi \varphi} Z(r, \eta), \quad (\text{C.11})$$

with  $Z(r, \eta)$  being an undetermined function. Substituting this ansatz into the master equation (3.15) yields the following partial differential equation for  $Z(r, \eta)$

$$\left[ (r + c) \partial_r^2 + Q_2^2 \partial_\eta^2 + \left( 3 + \frac{2c}{r} \right) \partial_r - Q_2^2 + \frac{1}{r} \right] Z(r, \eta) = 0. \quad (\text{C.12})$$

The general solution can again be found by separation of variables  $Z(r, \eta) = f(r)g(\eta)$ . The equation for  $g(\eta)$  is

$$Q_2^2 \partial_\eta^2 g = -\lambda g, \quad (\text{C.13})$$

for arbitrary values of  $\lambda$ . However, for non-zero values of  $\lambda$  the resulting solutions have either unphysical boundary conditions, or can be obtained by redefining the charges in (C.11). The remaining case is when  $\lambda = 0$  which leads to linear  $\eta$  dependence in  $Z(r, \eta)$  and does not affect the  $r$ -dependent factor  $f(r)$ .

For simplicity, we now focus on the class of solutions where  $Z(r, \eta) = Z(r)$  is  $\eta$ -independent. The  $\eta$ -dependence can trivially be restored. The exact solution of (C.12) is obtained by substituting

$$Z(r) = r^{-1} \tilde{Z}(\xi(r)), \quad \xi = r + c, \quad (\text{C.14})$$

into (C.12). This leads to a modified Bessel equation for  $\tilde{Z}(\xi)$

$$(\xi \partial_\xi^2 + \partial_\xi - Q_2^2) \tilde{Z}(\xi) = 0. \quad (\text{C.15})$$

Thus

$$Z(r) = C_1 r^{-1} I_0 (2|Q_2|\sqrt{r+c}) + C_2 r^{-1} K_0 (2|Q_2|\sqrt{r+c}) , \quad (C.16)$$

where  $I_0$  and  $K_0$  are modified Bessel functions of the second kind. The condition that the instanton corrections are exponentially suppressed for small string coupling requires that  $C_1 = 0$ . Hence we obtain the following exact two-charge membrane solution

$$\Lambda_1^{(2)} = \frac{C}{r} e^{iQ_\chi \chi + iQ_\varphi \varphi} K_0 \left( 2 \sqrt{Q_\chi^2 + Q_\varphi^2} \sqrt{r+c} \right) . \quad (C.17)$$

This solution corresponds to the membrane instanton corrections discussed in [14]. The  $\eta$ -dependence can be restored by replacing  $C \rightarrow C_1 + \eta C_2$ .

### C.2 Membrane instantons with fivebrane charge

Let us now proceed and discuss the case  $Q_\sigma \neq 0$ . This situation corresponds to a membrane instanton with charge  $Q_2$  which may also include a five-brane charge  $Q_\sigma$ . Starting point of the analysis are again the results of appendix A adapted to the membrane case by imposing (C.1) and (C.2). Considering the solution (A.14) with  $\tilde{Q}_5 = Q_\sigma$  we again have the two distinct cases of  $y = y(\chi, \varphi)$  and  $y = const$ , which will now be discussed in turn.

We first show that taking  $y = y(\chi, \varphi)$  non-constant is not compatible with the requirements (1) - (5). This can be seen from starting with the solution (A.22) and taking  $Q_5 = 0$

$$A_0 = \frac{C e^{iQ_\sigma (\sigma + \frac{1}{2} (\tan y (\chi^2 - F^2) - \int ((F')^2 - F^2) dy))}}{\left( \frac{\chi}{\cos y} + F \tan y + F' \right)^{1/2}} . \quad (C.18)$$

In the limit  $Q_5 = 0$  the divergent terms in (A.22) are absent. In order to rule out such solutions one has to proceed to the next non-trivial equation which is given by the real part of (A.19). Using

$$\varrho = \frac{\chi}{\cos y} + F \tan y + F', \quad \Delta_F = F' + \int F dy, \quad (C.19)$$

this equation becomes

$$2Q_\sigma \partial_\varrho A_{1/2}(\varrho, y) = Q_2^2 + Q_2 Q_\sigma (\varrho - \Delta_F) - \frac{Q_\sigma^2}{4} (12c + \Delta_F (2\varrho - \Delta_F)) + \frac{1}{4\varrho^4} (\varrho^2 - 2\Delta_F'' \varrho + 5(\Delta_F')^2) . \quad (C.20)$$

The solution of this equation has a singularity in the limit  $Q_\sigma \rightarrow 0$ . This contradicts the regularity condition (4). Thus this type of solutions is ruled out.

We then restrict ourselves to the case where  $y$  is constant. Using the U(1) isometry (2.8), one can perform a rotation in the  $\chi$ - $\phi$ -plane such that  $f_{1/2}$  depends on  $\chi$  only<sup>8</sup>

$$f_{1/2} = 2|Q_2| + |Q_\sigma| \chi, \quad (C.21)$$

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<sup>8</sup>We assume that we work in the region  $\chi > 0$ .

Adapting the result (A.25) then leads to

$$A_0 = C e^{iQ_\sigma \sigma + iQ_\varphi \varphi}. \quad (\text{C.22})$$

Subsequently  $A_{1/2}$  is determined by the real part of (A.19)

$$2|Q_\sigma| \partial_\chi A_{1/2} = (2Q_\sigma Q_\varphi + |Q_2| |Q_\sigma|) \chi - Q_\sigma^2 \left( \frac{3}{4} \chi^2 + 3c \right) + Q_2^2 - Q_\varphi^2. \quad (\text{C.23})$$

For  $Q_2$  and  $Q_\varphi$  independent, the last two terms in this equation will give rise to a singularity in  $A_{1/2}$  as  $Q_\sigma \rightarrow 0$  which would be in disagreement with our condition (4). The two possibilities to avoid this singularity are either taking  $Q_2 \propto Q_\sigma$  and  $Q_\varphi \propto Q_\sigma$  or identifying

$$Q_2^2 = Q_\varphi^2. \quad (\text{C.24})$$

Requiring that  $Q_2$  is an independent charge which does not vanish in the limit when  $Q_\sigma \rightarrow 0$  indicates that one should choose the condition (C.24). It is then straightforward to determine  $A_{1/2}$  by integrating (C.23),

$$A_{1/2} = -\frac{1}{8} |Q_\sigma| \chi^3 + \frac{2\epsilon + 1}{4} |Q_2| \chi^2 - \frac{3c}{2} |Q_\sigma| \chi + a_{1/2}(\varphi), \quad (\text{C.25})$$

where  $\epsilon = \text{sign}(Q_2 Q_\sigma)$  is the relative sign of the two charges and  $a_{1/2}$  is arbitrary function.

Subsequently, one moves to higher orders in the perturbative expansion. Imposing the reality of the  $A_k(\chi, \varphi)$ ,  $k > 0$ , each order of the expansion gives rise to two independent equations originating from the real and imaginary part of the coefficients. Using the equations coming from the imaginary part one can proof by induction that  $A_k(\chi, \varphi) = A_k(\chi)$ ,  $k > 0$  are independent of  $\varphi$  while the  $\chi$ -dependence of the  $A_k(\chi)$  is determined by the real parts of the expansion. In particular this result establishes that  $a_{1/2}(\varphi)$  appearing in (C.25) is constant.

The function  $A_1(\chi)$  is fixed by the differential equation at order  $r^{-1/2}$ . Upon substituting the previous results this equation becomes

$$2|Q_\sigma| \partial_\chi A_1 = -(\alpha + 2) |Q_\sigma| \chi - (2\alpha + 2 - \epsilon) |Q_2|. \quad (\text{C.26})$$

Similarly to the situation at the previous order the last term induces a singularity in  $Q_\sigma$  and should vanish separately. This fixes the parameter  $\alpha$  to

$$\alpha = \frac{\epsilon}{2} - 1. \quad (\text{C.27})$$

Integrating (C.26) then yields

$$A_1 = -\frac{2 + \epsilon}{8} \chi^2 + a_1. \quad (\text{C.28})$$

At this point it is illustrative to consider one additional order in the perturbative expansion, as this equation will give another restriction on the charges. Making use of the previous results the equation at order  $r^{-3/2}$  determines  $A_{3/2}(\chi)$  and reads

$$\begin{aligned} 2|Q_\sigma| \partial_\chi A_{3/2} = & \frac{5}{64} Q_\sigma^2 \chi^4 - \frac{1}{8} |Q_\sigma| |Q_2| (4\epsilon + 3) \chi^3 + \frac{3}{2} Q_2^2 (1 + \epsilon) \chi^2 + \frac{5}{8} Q_\sigma^2 c \chi^2 \\ & + |Q_\sigma| \left( \frac{1}{2} a_{1/2} - c |Q_2| (2 + 3\epsilon) \right) \chi + |Q_2| (a_{1/2} + c |Q_2|) \\ & - \frac{1}{4} (1 + \epsilon) + \frac{5}{4} c^2 Q_\sigma^2. \end{aligned} \quad (\text{C.29})$$

Requiring the absence of singularities in the limit  $Q_\sigma \rightarrow 0$  then requires

$$\epsilon = -1, \quad a_{1/2} = -c|Q_2|. \quad (\text{C.30})$$

Remarkably this implies that the solutions where  $Q_2$  and  $Q_\sigma$  have the same sign do not satisfy condition (4) and only solutions with  $\text{sign}(Q_2 Q_\sigma) = -1$  are physical.

Summarizing the results of the perturbative analysis we obtain

$$\begin{aligned} \Lambda_2^{(2)} = C r^{-3/2} e^{iQ_2\varphi+iQ_\sigma\sigma} \exp \left[ -(2|Q_2| + \chi|Q_\sigma|)\sqrt{r} \right. \\ \left. - (|Q_2|(\frac{1}{4}\chi^2 + c) + \frac{1}{2}\chi|Q_\sigma|(\frac{1}{4}\chi^2 + 3c))\sqrt{r}^{-1} + \dots \right], \end{aligned} \quad (\text{C.31})$$

where  $\text{sign}(Q_2 Q_\sigma) = -1$ .

We now use the input from the perturbative analysis to find an exact solution for the membrane-fivebrane case. The  $\varphi$ -independence of the coefficients  $A_k, k > 0$  thereby motivates considering the refined ansatz

$$\Lambda_2^{(2)} = e^{iQ_2\varphi+iQ_\sigma\sigma} Z(r, \chi). \quad (\text{C.32})$$

Substituting this ansatz into the master equation (3.15) leads to the following partial differential equation for  $Z(r, \chi)$

$$\left[ (r+c)\partial_r^2 + \partial_\chi^2 + \left(3 + \frac{2c}{r}\right)\partial_r - Q_\sigma^2 \left(r + \chi^2 + 3c + \frac{c^2}{r+c}\right) - Q_2^2 - 2\chi|Q_2 Q_\sigma| + \frac{1}{r} \right] Z = 0, \quad (\text{C.33})$$

where the condition  $\text{sign}(Q_2 Q_\sigma) = -1$  was used. As in the case of the fivebrane and membrane instanton corrections (B.17) and (C.12), equation (C.33) is separable and may be solved by separation of variables. Here we will, however, follow a different strategy and use input from the perturbative analysis (C.31) and eq. (C.33) to further restrict the ansatz (C.32). In this course we make the following observations. First notice that upon substituting  $Z(r, \chi) = r^{-1}\tilde{Z}(r, \chi)$ , (C.33) becomes a partial differential equation for  $\tilde{Z}(r, \chi)$  which depends on the variable  $\xi = r+c$  only. This implies that  $\tilde{Z}(r, \chi)$  depends on  $r$  through the combination  $r+c$ . Second one observes that the terms proportional to  $Q_2$  in (C.31) are the first two terms in the expansion of the instanton action (4.2) for large values of  $r$ , up to  $c$ -dependent terms which are generated by replacing  $r \rightarrow r+c$  in (4.2). Third the perturbative result (C.31) shows that the perturbative expansion around the instanton should start with  $r^{-3/2}$ . Investigating solutions of (C.33) which obey these restrictions one then arrives at the conclusion that  $Z(r, \chi)$  should be of the form

$$Z(r, \chi) = \frac{C}{r\sqrt{4(r+c) + \chi^2}} e^{-|Q_2|\sqrt{4(r+c)+\chi^2}-|Q_\sigma|f(r,\chi)}, \quad (\text{C.34})$$

where  $f(r, \chi)$  does not depend on the charges.

Substituting (C.34) into (C.33) leads to a differential equation for  $f(r, \chi)$ . This equation splits into three terms which are proportional to  $Q_\sigma^2$ ,  $Q_2 Q_\sigma$  and  $|Q_\sigma|$ , respectively.

Since we assumed  $f(r, \chi)$  to be charge-independent each of these terms has to vanish separately. This gives rise to the following three equations

$$\xi (\partial_\xi f)^2 + (\partial_\chi f)^2 = \xi + \chi^2 + 2c + \frac{c^2}{\xi}, \tag{C.35}$$

$$2\xi \partial_\xi f + \chi \partial_\chi f = \chi \sqrt{4\xi + \chi^2}, \tag{C.36}$$

$$(4\xi + \chi^2) (\partial_\chi^2 + \xi \partial_\xi^2) f + (\chi^2 \partial_\xi - 2\chi \partial_\chi) f = 0. \tag{C.37}$$

Remarkably these equations are simultaneously solved by taking

$$f = \frac{1}{2} \chi \sqrt{4\xi + \chi^2} - 2c \log \left[ \sqrt{\xi}^{-1} \left( \sqrt{4\xi + \chi^2} - \chi \right) \right]. \tag{C.38}$$

Thus the ansatz (C.34) indeed gives rise to a consistent solution of (C.33). The corresponding *exact* membrane instanton correction is based on the instanton action (4.2) and reads

$$\begin{aligned} \Lambda_2^{(2)} = C \frac{\left( \sqrt{r+c}^{-1} \left( \sqrt{4(r+c)+\chi^2} - \chi \right) \right)^{2c|Q_\sigma|}}{r \sqrt{4(r+c)+\chi^2}} e^{iQ_2 \varphi + iQ_\sigma \sigma} \times \\ \times \exp \left[ - \left( |Q_2| + \frac{1}{2} \chi |Q_\sigma| \right) \sqrt{4(r+c) + \chi^2} \right], \end{aligned} \tag{C.39}$$

where  $Q_2$  and  $Q_\sigma$  must have opposite signs and  $\chi > 0$ .

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